

## FREE SAMPLE: THE FIRST QUESTION

### Question One (8 marks)

(a) Solve  $z^3 = -8$  and give all three solutions exactly, in the form  $a + bi$ .

[2 marks]

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(b) Let  $\omega = \cos(\frac{2\pi}{5}) + i \sin(\frac{2\pi}{5})$ .

Explain why  $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ , and use it to show that  $\cos(\frac{2\pi}{5}) + \cos(\frac{4\pi}{5}) = \frac{-1}{2}$ .

[3 marks]

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(c) Hence find the exact value of  $\cos(\frac{2\pi}{5})$ . Show how you decide between the possible values.

[3 marks]

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## THE WORKED ANSWER TO THAT QUESTION

### Question 1 (8 marks) [Complex numbers and roots of unity]

(a) Solve  $z^3 = -8$  and give all three solutions exactly, in the form  $a + bi$ .

**(a) Answer:  $z = -2, z = 1 + i\sqrt{3}, z = 1 - i\sqrt{3}$**

- Entry step: worth marks in the 1-4 range of the 8-point question score.
- Write  $-8 = 8 \operatorname{cis}(\pi + 2k\pi)$ , so  $z = 2 \operatorname{cis}(\frac{\pi + 2k\pi}{3})$  for  $k = 0, 1, 2$ . [1 mark]
- $k = 0$ :  $1 + i\sqrt{3}$ ;  $k = 1$ :  $-2$ ;  $k = 2$ :  $1 - i\sqrt{3}$ . [1 mark]

#### KEY TAKEAWAY

Roots of a complex number: take the root of the modulus and divide the argument (plus  $2k\pi$ ) by  $n$ ; the  $n$  roots are evenly spaced around a circle.

(b) Let  $\omega = \cos(\frac{2\pi}{5}) + i \sin(\frac{2\pi}{5})$ . Explain why  $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ , and use it to show that  $\cos(\frac{2\pi}{5}) + \cos(\frac{4\pi}{5}) = -\frac{1}{2}$ .

**(b) Answer:  $\cos(\frac{2\pi}{5}) + \cos(\frac{4\pi}{5}) = -\frac{1}{2}$**

- Scholarship band (5-6): the part needs your own method and accurate algebra.
- By de Moivre  $\omega^5 = \operatorname{cis}(2\pi) = 1$ , so  $0 = \omega^5 - 1 = (\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4)$ . Since  $\omega \neq 1$ , the second factor is 0. [1 mark]
- Take real parts:  $1 + \cos(\frac{2\pi}{5}) + \cos(\frac{4\pi}{5}) + \cos(\frac{6\pi}{5}) + \cos(\frac{8\pi}{5}) = 0$ . [1 mark]
- $\cos(\frac{6\pi}{5}) = \cos(\frac{4\pi}{5})$  and  $\cos(\frac{8\pi}{5}) = \cos(\frac{2\pi}{5})$  (symmetry about the real axis), so  $1 + 2[\cos(\frac{2\pi}{5}) + \cos(\frac{4\pi}{5})] = 0$ , giving the result. [1 mark]

#### KEY TAKEAWAY

Sums of roots of unity vanish: multiply out  $(\omega - 1)$  times the geometric sum, then take real parts and use symmetry to pair the angles.

(c) Hence find the exact value of  $\cos(\frac{2\pi}{5})$ . Show how you decide between the possible values.

**(c) Answer:  $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5} - 1}{4}$**

- Outstanding band (7-8): the part needs insight and a complete, efficient argument.
- Let  $c = \cos(\frac{2\pi}{5})$ . Then  $\cos(\frac{4\pi}{5}) = 2c^2 - 1$  (double angle). Substituting into part (b):  $c + 2c^2 - 1 = -\frac{1}{2}$ , so  $4c^2 + 2c - 1 = 0$ . [1 mark]
- $c = (-2 \pm \sqrt{20}) \div 8 = \frac{-1 \pm \sqrt{5}}{4}$ . [1 mark]
- $\frac{2\pi}{5}$  is in the first quadrant, so  $c > 0$  and  $c = \frac{\sqrt{5} - 1}{4} \approx 0.309$ . (The other root is negative and belongs to  $\cos(\frac{4\pi}{5})$ .) [1 mark]

#### KEY TAKEAWAY

Two expressions for the same unknown: a double-angle identity turns the sum from the roots of unity into a quadratic you can solve exactly; use the quadrant to reject the wrong root.

LOS: Complex numbers and roots of unity (93202)